Fractionally charged Wilson loops as a probe of theta dependence in CP^{N-1} sigma models

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We present a method for exploring θ-dependence in two-dimensional $U(1)$ gauge theories which is based on the calculation of fractionally charged Wilson loops in the $\theta = 0$ theory. Including a Wilson loop with charge $q = \theta/2\pi$ in the path integral of a $U(1)$ gauge theory in two spacetime dimensions is equivalent to including a θ-term in the two-volume enclosed by the loop. We study the θ-dependence of the free energy density $\varepsilon(\theta)$ for the CP^1, CP^5 and CP^9 sigma models by extracting $\varepsilon(\theta)$ from the area law of a fractionally charged Wilson loop. For CP^1, $\varepsilon(\theta)$ is smooth in the region $\theta \approx \pi$ and well-described by a dilute instanton gas approximation throughout the range $0 < \theta < 2\pi$. For CP^5 and CP^9 the energy density exhibits a clear cusp and evidence for discrete degenerate vacua at $\theta = \pi$, as expected from large N arguments. For CP^9 the θ-dependence is in good quantitative agreement with the leading order large N prediction $\varepsilon(\theta) = \frac{1}{2}\chi_t\theta^2$ throughout the range $0 < \theta < \pi$.

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1. Background

The structure of topological charge in the QCD vacuum is central to an understanding of low-energy hadron dynamics. Lattice calculations have provided quantitative confirmation of the relation between topological susceptibility and the mass of the η' meson [1, 2]. The recent observation of extended, coherent, co-dimension 1 sheets of topological charge in Monte-Carlo generated SU(3) gauge configurations [3] points to a new paradigm for the QCD vacuum. Contrary to the standard instanton liquid picture, the vacuum is apparently a "topological sandwich" consisting of alternating sign membranes of topological charge. Longstanding arguments of Witten [4] based on large N chiral dynamics show that, at least at large N, instantons should disappear from the vacuum and be replaced by codimension-1 membranes which are in fact domain walls between discrete "k-vacua" where the effective local value of θ differs from that of the true ground state by $2\pi k$, with k an integer. The topological sandwich picture which has emerged from the Monte Carlo calculations is quite compatible with these large N arguments. This picture is in some ways a 4-dimensional analog of Coleman's picture of θ-dependence in the massive Schwinger model [5]. In this model, θ can be interpreted as a background electric field and the domain walls are just charged particles whose world lines have codimension 1. These world lines separate regions in which the electric flux differs by one unit, i.e. $\Delta\theta = \pm 2\pi$. A similar interpretation of θ-dependence applies to the two-dimensional CP^{N-1} sigma models, which also have a U(1) gauge invariance. Lüscher [6] clarified the analogy between 2D $U(1)$ theories and 4D Yang-Mills theory by pointing out the similar role played by the Chern-Simons tensors in the 2D and 4D theories. The Wilson loop in the CP^{N-1} model is analogous to a "Wilson bag" in QCD, which is an integral of the CS tensor of the Yang-Mills field over a three-dimensional surface. Monte Carlo studies of the CP^{N-1} models [7, 8] have shown that for $N > 3$ these models exhibit a topological structure quite analogous to that observed in lattice QCD, with the vacuum occupied by extended coherent topological charge structures of codimension 1 [7]. In striking contrast, the topological charge distributions in the CP^1 and CP^2 models are found to be dominated by small instantons [8]. The CP^{N-1} models thus provide a detailed example of Witten's arguments that instantons should "melt" or become irrelevant at large N. For these models, the instanton melting point is found to be about $N \approx 4$ [8]. (As discussed by Lüscher [9], the value of N at which instantons become irrelevant may depend on the lattice formulation, but for any latticization, at least the CP^1 model is expected to be dominated by small instantons.)

Although the value of the θ parameter in real-world QCD is zero to high accuracy, a deeper understanding of topological charge structure can by obtained by considering gauge theories with nonzero θ terms. Monte Carlo studies of QCD at nonzero θ are hampered by the fact that the θ term contributes an imaginary part to the Euclidean action. As a result, the exponentiated action in the path integral cannot be interpreted as a probablility, precluding the direct simulation of such theories by Monte Carlo techniques. Reweighting methods in which the $e^{i\theta\nu}$ factor in the path integral is introduced in the ensemble average over $\theta = 0$ configurations are sufficient for probing small values of θ. But some of the most interesting issues associated with multi-phase structure are most directly addressed by studying the region $\theta \approx \pi$, where such methods are ineffective. More sophisticated techniques for extending the reach of Monte Carlo techniques to large θ have shown promise, but definitive results near $\theta \approx \pi$ are still difficult to obtain by these methods.

2. Free energy of the θ-vacuum

We present a method for exploring θ-dependence in two-dimensional $U(1)$ gauge theories which is based on the calculation of fractionally charged Wilson loops in the $\theta = 0$ theory. The results for the free energy $\varepsilon(\theta) = E(\theta)/V$ in the CP^{N-1} models exhibit the power of this method, providing for the first time direct numerical evidence for a first order phase transition at $\theta = \pi$ for the $N > 4$ models. By contrast, the instanton dominated CP^1 model exhibits much smoother behavior in the $\theta = \pi$ region, as expected from a dilute instanton gas calculation. (Our results do not rule out a second order phase transition at $\theta = \pi$ in the CP^1 model, which is expected from theoretical arguments [10]).

The central observation which we exploit here is that, in the path integral for a 2D U(1) gauge theory, including a closed Wilson loop with charge $q = \theta/2\pi$ is equivalent to including a θ term in the two-volume enclosed by the loop. If the Wilson loop goes around the boundary ∂V of a two-volume V,

$$\theta \int_V d^2x Q(x) = \frac{\theta}{2\pi} \oint_{\partial V} A \cdot dx \tag{2.1}$$

where

$$Q(x) = \frac{1}{2\pi} \varepsilon_{\mu\nu} F^{\mu\nu} \tag{2.2}$$

is the topological charge density. Since the CP^{N-1} models have nonzero topological susceptibility, the free energy per unit volume $\varepsilon(\theta) = E(\theta)/V$ inside the loop is greater than $\varepsilon(0)$ outside. This gives rise to a linear confining potential between fractional charges or equivalently, an area law for large Wilson loops. The coefficient of the area law determines the difference in vacuum energy inside and outside the loop:

$$\langle W_{\mathscr{C}}(q) \rangle \sim \exp\left[-\left(\varepsilon(2\pi q) - \varepsilon(0)\right)V\right] \tag{2.3}$$

where $\mathscr{C} = \partial V$. Note that since $\varepsilon(\theta + 2\pi) = \varepsilon(\theta)$, the coefficient of the area law is periodic in the charge q. At integer values of charge, the confining force is completely screened and the area term in the Wilson loop vanishes. The behavior observed in the region $q \approx 1/2$ distinguishes between an instanton-dominated model and a large N domain wall model. For an instanton theory, a dilute gas approximation gives

$$\varepsilon(\theta) - \varepsilon(0) = \chi_t(1 - \cos\theta) \tag{2.4}$$

where χ_t is the topological susceptibility. On the other hand, large N considerations [4] predict quadratic behavior, with periodicity leading to cusps at odd multiples of π,

$$\varepsilon(\theta) - \varepsilon(0) = \frac{1}{2}\chi_t \min_{k \varepsilon \mathscr{Z}}(\theta - 2\pi k)^2 \tag{2.5}$$

Physically, these cusps represent "string-breaking" which occurs when it is energetically favorable to screen the background electric flux by one unit from $\frac{1}{2} + \delta$ to $-\frac{1}{2} + \delta$. For large loops, this can be interpreted as a first order phase transition taking place inside the loop.

3. Monte Carlo Results

We study fractionally charged Wilson loops for the CP^1, CP^5, and CP^9 models. As discussed in Ref. [8], CP^1 is instanton-dominated, while CP^5 and CP^9 are both above the instanton "melting point". It was found that the topological charge distribution for CP^1 is dominated, at large β, by small instantons, while for CP^5 and CP^9 instantons do not appear but topological charge instead appears in the form of coherent one-dimensional domain wall-like structures. In Ref. [8] results for the topological suscepibility were compared with the leading-order large N prediction,

$$\chi_t = \frac{3M^2}{\pi N} \tag{3.1}$$

where M is half the nonsinglet meson mass. Lattice calculation of χ_t using both the overlap and log-plaquette definitions of topological charge [11] show good agreement with (3.1) for CP^9. For each of the CP^{N-1} models the value of the coupling constant β was adjusted to give a correlation length of ≈ 5 to 7. [The CP^1, CP^5 and CP^9 data were obtained at $\beta =$1.2, 0.9 and 0.8 respectively, for which the meson masses are $\mu =$0.179(3), 0.186(3) and 0.212(2).]

The calculations were done on 50×50 and 100×100 lattices and the effect of finite volume was found to be negligible. By studying these three models, we cover the entire range of topological charge dynamics from CP^1 which is instanton-dominated, to CP^9 where topological structure is described with reasonable accuracy by the large N approximation. This assertion is further supported by the results for θ dependence of the vacuum energy presented in this paper. One of the most striking features of the Monte Carlo results for fractionally charged Wilson loops is the difference between the behavior in CP^1 versus that in CP^5 and CP^9 in the region $\pi < \theta < 2\pi$. Since $\varepsilon(\theta)$ is periodic and an even function of θ, its value in this region should be determined by its value in the range $0 < \theta < \pi$ by reflection around π, $\varepsilon(\theta) = \varepsilon(2\pi - \theta)$. As seen in Figure 1, the measured value of $\varepsilon(\theta)$ for CP^1 ($\times$'s) is nicely periodic and symmetric around $\theta = \pi$, and in fact fits well to the dilute instanton gas prediction (2.4) On the other hand, for CP^5 ($\square$'s) and CP^9 ($\circ$'s), the coefficient extracted from a simple area-law fit to the Wilson loops continues to rise beyond $\theta = \pi$, violating the expected symmetry. We will argue that the behavior of CP^5 and CP^9 for $\theta > \pi$ is an effect of having two nearly degenerate ground states.

This behavior can be easily understood in terms of the large N picture [4], in which there are two nearly degenerate quasi-vacua when $\theta \approx \pi$. One vacuum has a background electric field $\theta/2\pi \approx +\frac{1}{2}$. The other is the one in which a unit of flux has been screened so that $\theta/2\pi \approx -\frac{1}{2}$. A Wilson loop with length R in the spatial direction and T in the time direction can be interpreted as the T-dependent propagator of a "string" operator consisting of a $+q$ and a $-q$ charge separated by R with an amount $q = \theta/2\pi$ of electric flux between them. This state has a large overlap with the vacuum state containing background flux of $\theta/2\pi$. But for $\theta > \pi$, the true ground state is the one where the flux has been screened by one unit to $\theta/2\pi - 1$. In order for the Wilson line to couple to this screened vacuum, the flux string must break via vacuum polarization. It is thus expected that, for $\theta > \pi$, the Wilson line will have a much larger overlap with the false (unscreened) vacuum than with the true (screened) vacuum. The Wilson loop area law thus tends to be determined by the energy of the unscreened vacuum, even for $\theta > \pi$ where the screened vacuum has lower energy. Since the Wilson line couples preferentially to the unscreened vacuum, we expect that our results
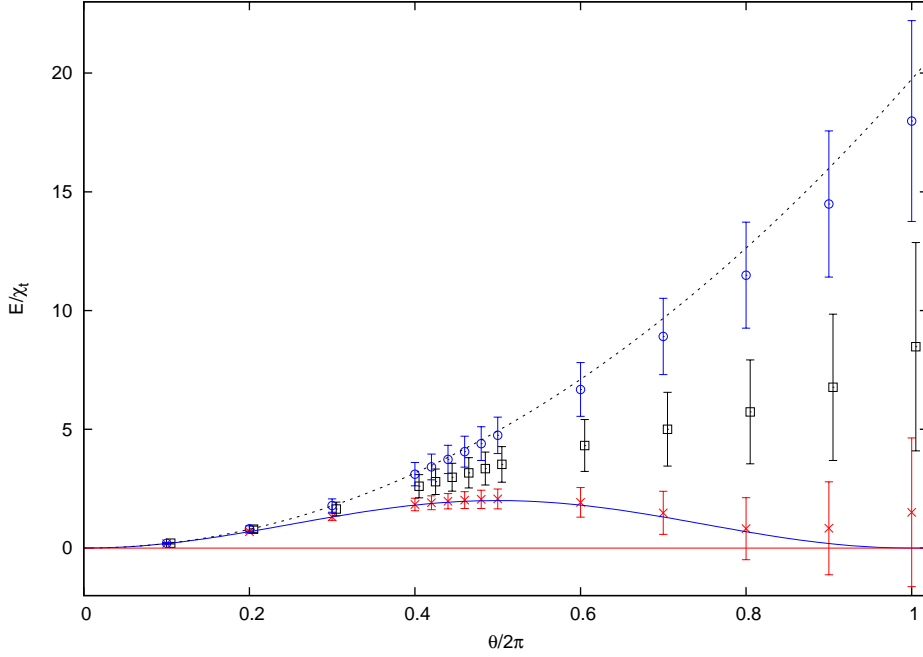
Figure 1: The free energy density $\varepsilon(\theta)$ for CP^1 ($\times$'s), CP^5 ($\square$'s) and CP^9 ($\circ$'s) extracted from fractionally charged Wilson loops. The lower and upper curves are the instanton gas and large N predictions, normalized to the same topological susceptibility. Note that, for $\theta/2\pi > \frac{1}{2}$ the large N curve is interpreted as the energy of the false (unscreened) vacuum.

for $\varepsilon(\theta)$ are measuring the true ground state energy throughout the range $0 < \theta < \pi$, where the unscreened vacuum is the true vacuum. By invoking the reflection symmetry about π we obtain a complete determination of $\varepsilon(\theta)$.

The results for $\varepsilon(\theta)$ for CP^1 are plotted in Figure 2. With the topological susceptibility fixed to the value obtained from the fluctuation of the integer-valued global topological charge v, $\chi_t = \langle v^2 \rangle / V$, the solid line is a zero-parameter fit to the dilute instanton gas formula (2.4). Also plotted (dotted line) is the leading order large N prediction, $\varepsilon(\theta) - \varepsilon(0) = \frac{1}{2} \chi_t \theta^2$. The corresponding results for CP^5 and CP^9 are shown in Figures 3 and 4. The solid curve is again the instanton gas prediction with normalization fixed to the measured χ_t and the dotted line is the normalized large N prediction. Unlike the CP^1 case, the data shows a clear departure from the instanton gas model in the direction of the large N prediction.

In particular, there is convincing evidence that $\varepsilon(\theta)$ has a positive slope at $\theta = \pi$, and since $\varepsilon(\theta) = \varepsilon(2\pi - \theta)$, this implies a cusp with discontinuous derivative at $\theta = \pi$. Moreover, the results for $\varepsilon(\theta)$ for the CP^9 model are in good agreement with the large N prediction (2.5) in both the magnitude of χ_t and in the shape of the function $\varepsilon(\theta)$. In all of the models sudied, the value of χ_t obtained from fractionally charged Wilson loops using $\chi_t = \varepsilon''(\theta)$ is in excellent agreement with that obtained from the fluctuations of the integer-valued global topological charge. This agreement provides evidence that the fractionally charged Wilson loop method gives an accurate determination of $\varepsilon(\theta)$ over the entire range $0 < \theta < \pi$.
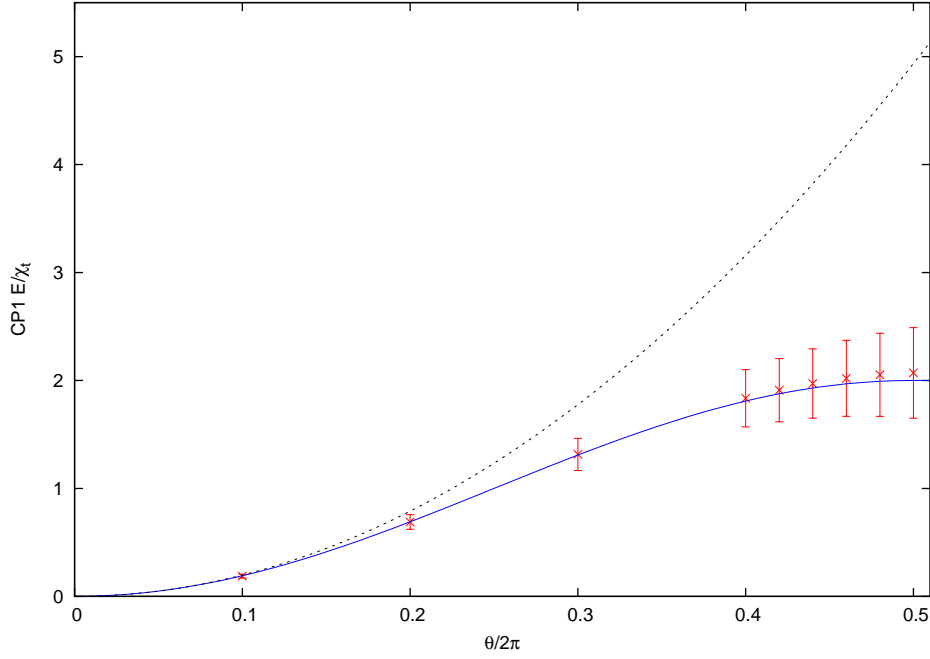
Figure 2: $\varepsilon(\theta)$ for the CP^1 model. The lower (solid) and upper (dotted) curves are the instanton gas and large N predictions, normalized to the same topological susceptibility.
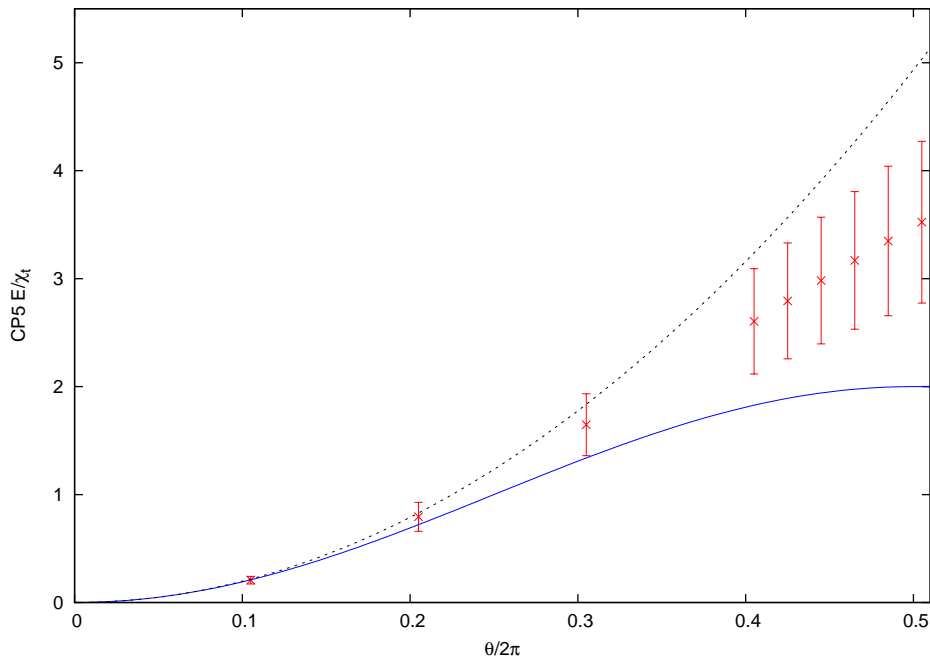

Figure 3: $\varepsilon(\theta)$ for the CP^5 model. The lower (solid) and upper (dotted) curves are the instanton gas and large N predictions, normalized to the same topological susceptibility.
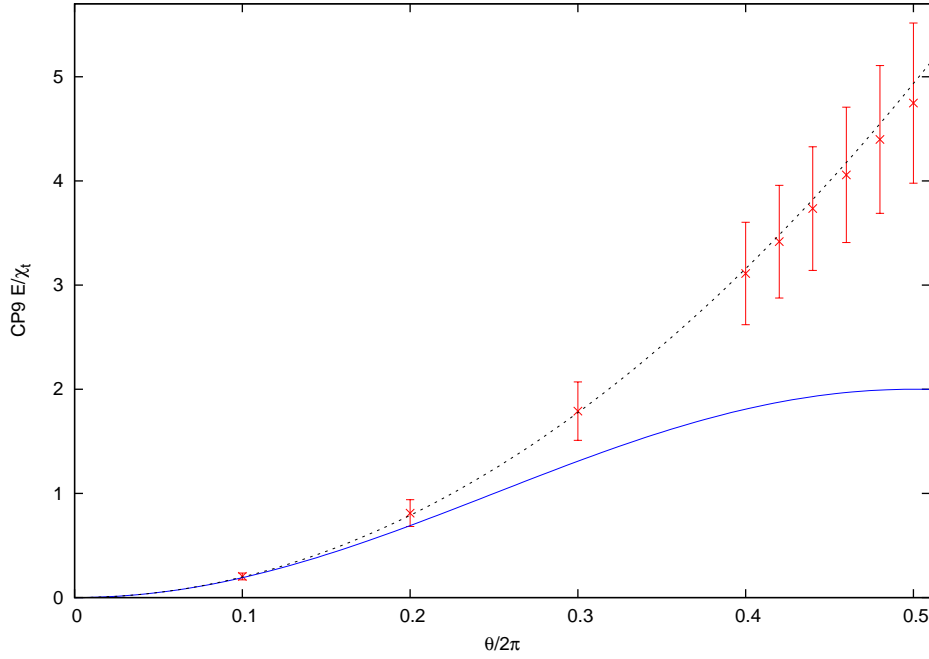
Figure 4: $\varepsilon(\theta)$ for the CP^9 model. The lower (solid) and upper (dotted) curves are the instanton gas and large N predictions, normalized to the same topological susceptibility.

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